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On fuzzy S^*N spaces

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Abstract. In this paper, a new class of fuzzy topological spaces namely fuzzy S^*N-spaces is introduced by means of fuzzy simply* open sets. Several characterizations of fuzzy S^*N-spaces are obtained. The inter-relationships between fuzzy normal spaces, fuzzy semi normal spaces and fuzzy S^*N-spaces are established. The conditions, under which fuzzy ∂^* spaces, fuzzy D-Baire spaces become fuzzy S^*N-spaces, are obtained in this paper.

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1. Introduction

The universe is a complex system filled with uncertainties and problems of vagueness have probably always existed in human experience. Vagueness is not regarded with suspicion, but is simply an acknowledged characteristic of the world around us. Mathematics equips us with the tools to quantify and manage these uncertainties. Human concepts have a graded structure in that whether or not a concept applies to a given object is a matter of degree, rather than a yes - or - no question, and that people are capable of working with the degrees in a consistent way. In 1965, Zadeh [1] in his classic paper, called the concepts with a graded structure "fuzzy concepts "and proposed the notion of a "fuzzy set "that gave birth to the field of fuzzy logic. The potential of fuzzy notion was realized by the researchers and has successfully been applied for new investigations in all the branches of science and technology for more than last five decades. In 1968, Chang [2] introduced the concept of fuzzy topological space. After that various works on fuzzy topological spaces have been developed in various directions. In the recent years, there has been a growing trend to introduce and study different forms of fuzzy topological spaces.

Normality is one of the few separation axioms which can be defined purely in terms of the properties of the open and closed sets. Hutton [3] defined the notion of normality in fuzzy topological spaces. There are many articles on fuzzy normal topological spaces which were stated by many authors like Wuyts and Lowen [4], Ali [5] and other researchers. In 2016, Majeed and AL-Bayati [6] defined and studied the notion of S^*N-spaces by means of simply open sets and investigated their relationship to other topological spaces. The notion of fuzzy simply* open sets in fuzzy topological spaces was introduced and studied by Thangaraj and Dinakaran in [7] by means of fuzzy open sets and fuzzy residual sets. Residuals often refer to operations or structures derived from logical connectives in fuzzy logic, particularly those related to implications. Residual implications (or residuated implications) arise in the context of residuated lattices, which generalize classical logical operations for fuzzy sets. Residual sets can help in reasoning about partial truths in fuzzy systems. They are useful for defining rules in systems where precise thresholds are unavailable. Residual implications are used in approximate reasoning and inference mechanisms.

In this paper, the notion of fuzzy S^*N spaces is introduced by means of fuzzy simply* open sets and studied. Several characterizations of fuzzy S^*N spaces are obtained. The inter-relationships between fuzzy normal spaces, fuzzy seminormal spaces and fuzzy S^*N spaces are established. The conditions, under which fuzzy ∂^* spaces, fuzzy D-Baire spaces become fuzzy S^*N-spaces, are also obtained in this paper.

In recent years, the topological space theory has been embedding in the soft set theory to obtain some interesting applications [8, 9]. Many authors [10, 11] redefined the classical topological concepts via soft topological structure. Recently, Şenel [12] applied the concept of octahedron sets proposed by Lee et al. [13, 14] to multi-criteria group decision making problems. Moreover Lee et al. [15] provided an insight into a cubic crisp sets and their applications to topology. On these lines, there is a need and scope of investigation considering different types of fuzzy normality, for applying some fuzzy topological concepts to information science and decision-making problems.

2. Preliminaries

Some basic notions and results used in the sequel, are given in order to make the exposition self - contained. In this work by (X, T) or simply by X, we will denote a fuzzy topological space due to Chang (1968). Let X be a non-empty set and I the unit interval [0,1]. A fuzzy set λ in X is a mapping from X into I. The fuzzy set 0_X is defined as $0_X(X) = 0$ for all $x \in X$ and the fuzzy set 1_X is defined as $1_x(x) = 1$ for all $x \in X$. For any fuzzy set λ in X and a family $(\lambda_i)_{i\in j}$ of fuzzy set in X, the *compliment* $\lambda^{'}$, the *union* $\bigvee_{i\in J} \lambda_i$ and the *intersection* $\bigwedge_{i\in J} \lambda_i$ are defined respectively as follows: for each $x \in X$, $\lambda^{'}(x) = 1-\lambda(x)$, $(\bigvee_{i\in J} \lambda_i)(x) = sup_{i\in J}\lambda_i(x)$, $(\bigwedge_{i\in j} \lambda_i)(x) = inf_{i\in J}\lambda_i(x)$, where J is an index set.

Definition 2.1 ([2]). A *fuzzy topology* on a set X is a family T of fuzzy sets in X which satisfies the following conditions:
(i) $0_X \in T$, $1_X \in T$,
(ii) if $A, B \in T$, then $A \wedge B \in T$,
(iii) If $A_i \in T$ for each $i \in J$, then $\bigvee_i A_i \in T$.
The pair (X, T) is called a *fuzzy topological space*. Members of T are called *fuzzy open sets* in X and their complements are called *fuzzy closed sets* in X.

Definition 2.2 ([2]). Let (X, T) be a fuzzy topological space and λ be any fuzzy set in X. Then the *interior* and the *closure* of λ are defined respectively as follows:
(i) $int(\lambda) = \bigvee\{\mu : \mu \leq \lambda, \mu \in T\}$,
(ii) $cl(\lambda) = \bigwedge \{\mu' : \lambda \leq \mu', \mu \in T\}$.

Lemma 2.3 ([16]). *For a fuzzy set λ of a fuzzy topological space X,*
(1) $1 - int(\lambda) = cl(1 - \lambda)$,
(2) $1 - cl(\lambda) = int(1 - \lambda)$.

Definition 2.4. A fuzzy set λ in a fuzzy topological space (X, T) is called a

(i) *fuzzy regular-open set* in X, if $\lambda = intcl(\lambda)$ and *fuzzy regular-closed set* in X, if $\lambda = clint(\lambda)$ [16],
(ii) *fuzzy G_δ-set* in X, if $\lambda = \bigwedge_{i=1}^{\infty} \lambda_i$, where $\lambda_i \in T$ for $i \in I$ and *fuzzy F_σ-set* in X, if $\lambda = \bigvee_{i=1}^{\infty} \lambda_i$, where $1 - \lambda_i \in T$ for $i \in I$ [17],
(iii) *fuzzy dense set* in X, if there exists no fuzzy closed set μ in (X, T) such that $\lambda < \mu < 1$, i.e., $cl(\lambda) = 1$ [18],
(iv) *fuzzy nowhere dense set* in X, if there exists no non-zero fuzzy open set μ in X such that $\mu < cl(\lambda)$, i.e., $intcl(\lambda) = 0$ [18],
(v) *fuzzy somewhere dense set* in X, if there exists a non-zero fuzzy open set μ in (X, T) such that $\mu < cl(\lambda)$, i.e., $intcl(\lambda) \neq 0$ [19],
(vi) *fuzzy first category set* in X, if $\lambda = \bigvee_{i=1}^{\infty} \lambda_i$, where each λ_i is a fuzzy nowhere dense set in X. Any other fuzzy set in X is said to *be of fuzzy second category* [18],
(vii) *fuzzy residual set* in X, if $1 - \lambda$ is a fuzzy first category set in X [20],
(viii) *fuzzy σ-nowhere dense set* in X,) if λ is a fuzzy F_σ-set with $\mathrm{int}(\lambda) = 0$ [21],
(ix) *fuzzy σ-boundary set* in X, if $\lambda = \bigvee_{i=1}^{\infty} \mu_i$, where $\mu_i = \mathrm{cl}(\lambda_i) \wedge (1 - \lambda_i)$ and each λ_i is a fuzzy regular open sets in X [22],
(x) *fuzzy pseudo-open set* in X if $\lambda = \mu \vee \delta$, where μ is a non-zero fuzzy open set in X and δ is a fuzzy first category set in X [23].

Definition 2.5. A fuzzy topological space (X, T) is called a

(i) *fuzzy extremely disconnected space*, if the closure of each fuzzy open set of X is fuzzy open in X [24],
(ii) *fuzzy nodef space*, if each fuzzy nowhere dense set is a fuzzy F_σ-set in X [25],
(iii) *fuzzy ∂^* space*, if each fuzzy G_δ-set in X is a fuzzy simply* open set in X [26],
(iv) *fuzzy O_z-space*, if each fuzzy regular closed set is a fuzzy G_δ-set in X [27],
(v) *weak fuzzy O_z-space*, if for each fuzzy F_σ-set δ in X, $\mathrm{cl}(\delta)$ is a fuzzy G_δ-set in X [27],

(vi) *fuzzy quasi-O_z space*, if for a fuzzy regular closed set λ in X, there exists a fuzzy G_δ-set μ in X such that $\lambda = \mathrm{cl}\,\mathrm{int}(\mu)$ [28],
(vii) *fuzzy normal space*, if for each pair of fuzzy closed sets C_1 and C_2 such that $C_1 \leq 1 - C_2$, there exist fuzzy open sets M_1 and M_2 such that $C_i \leq M_i$ $(i = 1, 2)$ and $M_1 \leq 1 - M_2$ [3],
(viii) *fuzzy globally disconnected space*, if each fuzzy semi-open set in X is fuzzy open in X [29],
(ix) *fuzzy Baire space*, if $\mathrm{int}\left(\bigwedge_{i=1}^{\infty} \lambda_i\right) = 0$, where each λ_i is a fuzzy nowhere dense sets in X [20],
(x) *fuzzy D-Baire space*, if every fuzzy first category set in X is a fuzzy nowhere dense set in X [30],
(xi) *fuzzy perfectly disconnected space*, if for any two non-zero fuzzy sets λ and μ in X with $\lambda \leq 1 - \mu$, $\mathrm{cl}(\lambda) \leq 1 - \mathrm{cl}(\mu)$ [31],
(xii) *fuzzy semi normal space*, if for a fuzzy closed set λ and a fuzzy open set μ in X such that $\lambda \leq \mu$, there exists a fuzzy regular open set σ such that $\lambda \leq \sigma \leq \mu$ [32].

Theorem 2.6 ([16]). *In a fuzzy topological space,*
(1) *the closure of a fuzzy open set is a fuzzy regular closed set,*
(2) *the interior of a fuzzy closed set is a fuzzy regular open set.*

Theorem 2.7 ([33]). *If λ is a fuzzy simply* open set in a fuzzy topological space (X, T) in which fuzzy nowhere dense sets are F_σ-sets, then λ is a fuzzy pseudo - open set in X.*

Theorem 2.8 ([33]). *If λ is a fuzzy pseudo-open set in a fuzzy D-Baire space (X, T), then λ is a fuzzy simply* -open set in X.*

Theorem 2.9 ([31]). *If λ is a fuzzy nowhere dense set in a fuzzy perfectly disconnected space (X, T), then$\lambda = 0$.*

Theorem 2.10 ([27]). *If μ is a fuzzy regular closed set in a fuzzy extremally disconnected space (X, T), then μ is a fuzzy open G_δ -set in X.*

Theorem 2.11 ([7]). *If λ is a fuzzy simply* open set in a fuzzy topological space (X, T), then int$(\lambda) \neq 0$.*

Theorem 2.12 ([26]). *If μ is a fuzzy co-σ-boundary set in a fuzzy ∂^* space (X, T), then μ is a fuzzy simply* open set in X.*

Theorem 2.13 ([26]). *If μ is a fuzzy residual set in a fuzzy globally disconnnected and fuzzy ∂^* space (X, T), then μ is a fuzzy simply* open set in X.*

Theorem 2.14 ([20]). *Let (X, T) be a fuzzy topological space. Then the following are equivalent:*

(1) *(X, T) is a fuzzy weakly Baire space,*
(2) *int$(\lambda) = 0$ for each fuzzy first categary set λ in X,*
(3) *cl$(\mu) = 1$ for each fuzzy residual set μ in X.*

3. Fuzzy simply* open sets

Definition 3.1. A fuzzy set λ in a fuzzy topological space (X, T) is called a *fuzzy simply* open set* in X, if $\lambda = \mu \vee \delta$, where μ is a fuzzy open set and δ is a fuzzy nowhere dense set in X and $1 - \lambda$ is called a *fuzzy simply* closed set* in X.

Example 3.2. Let $X = \{a, b, c\}$. Consider the fuzzy sets λ, μ and γ defined on X as follows:

$$\lambda(a) = 0.7,\ \lambda(b) = 0.5,\ \lambda(c) = 0.6,$$
$$\mu(a) = 0.4,\ \mu(b) = 0.7,\ \mu(c) = 0.5,$$
$$\gamma(a) = 0.6,\ \gamma(b) = 0.4,\ \gamma(c) = 0.8.$$

Then $T = \{0, \lambda, \mu, \gamma, \lambda \vee \mu, \lambda \vee \gamma, \mu \vee \gamma, \lambda \wedge \mu, \lambda \wedge \gamma, \mu \wedge \gamma, \mu \vee [\lambda \wedge \gamma], \gamma \vee [\lambda \wedge \mu], \lambda \wedge [\mu \vee \gamma], \lambda \vee \mu \vee \gamma, 1\}$ is a fuzzy topology on X.

By computation, one can find that $\lambda, \mu, \gamma, \lambda \vee \mu, \lambda \vee \gamma, \mu \vee \gamma, \lambda \wedge \gamma, \mu \vee [\lambda \wedge \gamma], \gamma \vee [\lambda \wedge \mu], \lambda \wedge [\mu \vee \gamma]$ and $\lambda \vee \mu \vee \gamma$, are fuzzy dense sets in X and

$$cl(\lambda \wedge \mu) = 1 - (\lambda \wedge \mu),\ cl(\mu \wedge \gamma) = 1 - (\lambda \wedge \mu).$$

Also, $\int\{1 - (\lambda \wedge \mu)\} = \lambda \wedge \mu; \int\{1 - (\mu \wedge \gamma)\} = \lambda \wedge \mu$. The fuzzy nowhere dense sets in X are $1 - \lambda, 1 - \mu, 1 - \gamma, 1 - (\lambda \vee \mu), 1 - (\lambda \vee \gamma), 1 - (\mu \vee \gamma), 1 - (\lambda \wedge \gamma), 1 - (\mu \vee [\lambda \wedge \gamma]), 1 - \{\gamma \vee [\lambda \wedge \mu]\}, 1 - \{\lambda \wedge [\mu \vee \gamma]\}$ and $1 - (\lambda \vee \mu \vee \gamma)$.

By computation, one can find that $\lambda, \mu, \gamma, \lambda \vee \mu, \lambda \vee \gamma, \mu \vee \gamma, \lambda \wedge \mu, \lambda \wedge \gamma, \mu \wedge \gamma, \mu \vee [\lambda \wedge \gamma], \gamma \vee [\lambda \wedge \mu], \lambda \wedge [\mu \vee \gamma], (\lambda \vee \mu \vee \gamma)$ and $\mu \vee (1 - \mu), \{(\lambda \wedge \mu) \vee (1 - \mu)\}, \{(\mu \wedge \gamma) \vee (1 - \mu)\}, \{\lambda \vee (1 - \gamma)\}, \{\gamma \vee (1 - \gamma)\}, \{(\lambda \vee \gamma) \vee (1 - \gamma)\}, \{(\lambda \wedge \mu) \vee (1 - \gamma)\}, \{(\lambda \wedge \gamma) \vee (1 - \gamma)\}$ are fuzzy simply* open sets in X.

It should be noted that the fuzzy regular open set in X is $\lambda \wedge \mu$, since $intcl(\lambda \wedge \mu) = int(1 - (\lambda \wedge \mu)) = \lambda \wedge \mu$.

Proposition 3.3. *If a fuzzy set λ is a fuzzy simply* open set in a fuzzy topological space (X, T) , then there exists a fuzzy regular closed set θ in X such that $\theta \leq clint(\lambda)$.*

Proof. Suppose λ is a fuzzy simply* open set in X. Then $\lambda = \mu \vee \delta$, where μ is a fuzzy open set and δ is a fuzzy nowhere dense set in X. Since δ is a fuzzy nowhere dense set in X, $intcl(\delta) = 0$ and $int(\delta) \leq intcl(\delta)$ implies that $int(\delta) = 0$. Now $clint(\lambda) = clint(\mu \vee \delta) \geq clint(\mu) \vee clint(\delta) = cl(\mu) \vee cl(0) = cl(\mu)$. By Theorem 2.6, $cl(\mu)$ is a fuzzy regular closed set in X. Let $\theta = cl(\mu)$. Then for a fuzzy simply* open set λ in X, there exists a fuzzy regular closed set θ such that $\theta \leq clint(\lambda)$. □

Proposition 3.4. *If a fuzzy set λ is a fuzzy simply* open set in a fuzzy topological space (X, T), then $cl(\lambda)$ is a fuzzy F_σ-set in X.*

Proof. Suppose λ is a fuzzy simply* open set in X. Then by Proposition 3.3, there exists a fuzzy regular closed set θ in X such that $\theta \leq clint(\lambda)$. Since a fuzzy regular closed set is a fuzzy closed set in X, θ is a fuzzy closed set in X. Now $\theta \leq clint(\lambda) \leq cl(\lambda)$ implies that $cl(\lambda) = \theta \vee clint(\lambda) \vee cl(\lambda)$. Thus $cl(\lambda)$ is a fuzzy F_σ-set in X. □

Proposition 3.5. *If a fuzzy set λ is a fuzzy simply* open set in a fuzzy topological space (X, T), then $cl(\lambda)$ is not a fuzzy σ-nowhere dense set in X.*

Proof. Suppose λ is a fuzzy simply* open set in X. Then $\lambda = \mu \vee \delta$, where μ is a fuzzy open set and δ is a fuzzy nowhere dense set in X. By Proposition 3.4, $cl(\lambda)$ is a fuzzy F_σ-set in X. Now we have

$$intcl(\lambda) = intcl(\mu \vee \delta) \geq intcl(\mu) \vee intcl(\delta) = intcl(\mu) \vee 0 = intcl(\mu) \geq \mu \neq 0.$$

Thus $cl(\lambda)$ is a fuzzy F_σ-set in X with $intcl(\lambda) \neq 0$. So $cl(\lambda)$ is not a fuzzy σ-nowhere dense set in X. □

Proposition 3.6. *If a fuzzy set μ is a fuzzy simply* closed set in a fuzzy topological space (X,T), then $int(\mu)$ is a fuzzy G_δ-set in X.*

Proof. Suppose μ is a fuzzy simply* closed set in X. Then $1-\mu$ is a fuzzy simply* open set in X. Then by Proposition 3.4, $cl(1-\mu)$ is a fuzzy F_σ-set in X. Thus by Lemma 2.3, $1 - int(\mu) = cl(1-\mu)$. So $1 - int(\mu)$ is a fuzzy F_σ-set in X. Hence $int(\mu)$ is a fuzzy G_δ-set in X. □

Proposition 3.7. *If a fuzzy set λ is a fuzzy simply* open set in a fuzzy O_z-space (X,T), then there exists a fuzzy G_δ-set θ in X such that $\theta \leq clint(\lambda)$.*

Proof. Suppose λ is a fuzzy simply* open set in X. Then by Proposition 3.3, there exists a fuzzy regular closed set θ in X such that $\theta \leq clint(\lambda)$. Since X is a fuzzy O_z-space, the fuzzy regular closed set θ is a fuzzy G_δ-set in X. Thus for a fuzzy simply* open set λ in X, there exists a fuzzy G_δ-set θ in X such that $\theta \leq clint(\lambda)$. □

Remark 3.8. From proposition 3.7, one will have the following result:
If λ is a fuzzy simply* open set in a fuzzy O_z-space (X,T), then there exists a fuzzy G_δ-set θ in (X,T) such that $\theta \leq cl(\lambda)$, where $cl(\lambda)$ is a fuzzy F_σ-set in X.

Proposition 3.9. *If a fuzzy set λ is a fuzzy simply* open set in a fuzzy quasi-O_z-space (X,T), then there exists a fuzzy G_δ-set μ in X such that $clint(\mu) \leq clint(\lambda)$.*

Proof. Suppose λ is a fuzzy simply* open set in X. Then by Proposition 3.3, there exists a fuzzy regular closed set θ in X such that $\theta \leq clint(\lambda)$. Since X is a fuzzy quasi-O_z-space, for the fuzzy regular closed set θ, there exists a fuzzy G_δ-set μ in X such that $\theta = clint(\mu)$. Thus it follows that $clint(\mu) \leq clint(\lambda)$. □

The following proposition shows that fuzzy simply open sets coincide with fuzzy open sets in fuzzy perfectly disconnected spaces.*

Proposition 3.10. *If a fuzzy set λ is a fuzzy simply* open set in a fuzzy perfectly disconnected space (X,T), then λ is a fuzzy open set in X.*

Proof. Suppose λ is a fuzzy simply* open set in X. Then $\lambda = \mu \vee \delta$, where μ is a fuzzy open set and δ is a fuzzy nowhere dense set in X. Since X is a fuzzy perfectly disconnected space, by Theorem 2.9, $\delta = 0$. Thus $\lambda = \mu$. So λ is a fuzzy open set in X. □

Proposition 3.11. *If a fuzzy set λ is a fuzzy simply* open set in a fuzzy extremally disconnected space (X,T), then there exists a fuzzy open G_δ-set θ in X such that $\theta \leq clint(\lambda)$.*

Proof. Suppose λ is a fuzzy simply* open set in X. Then by Proposition 3.3, there exists a fuzzy regular closed set θ in X such that $\theta \leq clint(\lambda)$. Since (X, T) is a fuzzy extremally disconnected space, by Theorem 2.10, the fuzzy regular closed set θ is a fuzzy open G_δ-set in X. So for the fuzzy simply* open set λ, there exists a fuzzy open G_δ-set θ in X such that $\theta \leq clint(\lambda)$. □

Corollary 3.12. *If a fuzzy set λ is a fuzzy simply* open set in a fuzzy extremally disconnected space (X, T), then there exists a fuzzy G_δ-set θ in X such that $\theta \leq cl(\lambda)$.*

Proposition 3.13. *If a fuzzy set μ is a fuzzy simply* closed set in a fuzzy extremally disconnected space (X, T), then there exists a fuzzy closed F_σ-set ρ in X such that $intcl(\mu) \leq \rho$.*

Proof. Suppose μ is a fuzzy simply* closed set in X. Then $1 - \mu$ is a fuzzy simply* open set in X. Thus by Proposition 3.11, there exists a fuzzy open G_δ-set θ in X such that $\theta \leq clint(1 - \mu)$. So $\theta \leq 1 - intcl(\mu)$ and $intcl(\mu) \leq 1 - \theta$. Let $\rho = 1 - \theta$. Then ρ is a fuzzy closed F_σ-set in X and $intcl(\mu) \leq \rho$. □

Corollary 3.14. *If a fuzzy set μ is a fuzzy simply* closed set in a fuzzy extremely disconnected space (X, T), then there exists a fuzzy F_σ-set ρ in X such that $int(\mu) \leq \rho$.*

Proposition 3.15. *If $\lambda_1 \leq 1 - \lambda_2$, where λ_1 and λ_2 are fuzzy simply* open sets in a fuzzy topological space (X, T), then $cl(\lambda_1) \neq 1$ and $cl(\lambda_2) \neq 1$.*

Proof. Suppose $\lambda_1 \leq 1 - \lambda_2$, for the fuzzy simply* open sets λ_1 and λ_2 in X. Then $int(\lambda_1) \leq int(1 - \lambda_2)$. By Lemma 2.3, $int(1 - \lambda_2) = 1 - cl(\lambda_2)$. Thus $int(\lambda_1) \leq 1 - cl(\lambda_2)$. By Theorem 2.11, for the fuzzy simply* open set λ_1 in X, $int(\lambda_1) \neq 0$. So $1 - cl(\lambda_2) > 0$. Hence $cl(\lambda_2) \neq 1$.

Also, $\lambda_1 \leq 1 - \lambda_2$ implies that $\lambda_2 \leq 1 - \lambda_1$. Then $int(\lambda_2) \leq int(1 - \lambda_1) = 1 - cl(\lambda_1)$ and $int(\lambda_2) \neq 0$. Thus $int(1 - \lambda_1) \neq 0$. So $cl(\lambda_1) \neq 1$. □

Corollary 3.16. *If λ_1 and λ_2 are disjoint fuzzy simply* open sets in a fuzzy topological space (X, T), then $cl(\lambda_1) \neq 1$ and $cl(\lambda_2) \neq 1$.*

Proof. Suppose λ_1 and λ_2 are disjoint fuzzy simply* open sets in X. Then $\lambda_1 \wedge \lambda_2 = 0$. Thus $\lambda_1 \leq 1 - \lambda_2$. So by Proposition 3.15, for the fuzzy simply* open sets λ_1 and λ_2 in X, $\mathrm{cl}(\lambda_1) \neq 1$ and $\mathrm{cl}(\lambda_2) \neq 1$. □

Proposition 3.17. *If a fuzzy set λ is a fuzzy G_δ-set in a fuzzy ∂^* space (X, T), then $cl(\lambda)$ is a fuzzy F_σ-set in X.*

Proof. Suppose λ is a fuzzy G_δ-set in X. Then λ is a fuzzy simply* open set in X. By Proposition 3.4, for the fuzzy simply* open set λ, $cl(\lambda)$ is a fuzzy F_σ-set in X. □

Corollary 3.18. *If a fuzzy set μ is a fuzzy F_σ-set in a fuzzy ∂^* space (X, T), then $int(\mu)$ is a fuzzy G_δ-set in X.*

Proposition 3.19. *If λ is a fuzzy residual set in a fuzzy globally disconnected and fuzzy ∂^* space (X, T), then $cl(\lambda)$ is a fuzzy F_σ-set in X.*

Proof. Suppose λ is a fuzzy residual set in X. Since X is a fuzzy globally disconnected and fuzzy ∂^*, by Theorem 2.13, λ is a fuzzy simply* open set in X. Then by Proposition 3.4, $cl(\lambda)$ is a fuzzy F_σ-set in X. □

Remark 3.20. From Proposition 3.19, one will have the following result:
If λ is a fuzzy residual set in a fuzzy globally disconnected and fuzzy ∂^* space (X, T), then there exists a fuzzy F_σ-set η in X such that $\lambda \leq \eta$.

Corollary 3.21. *If a fuzzy set μ is a fuzzy first category set in a fuzzy ∂^* space (X, T), then $int(\mu)$ is a fuzzy G_δ-set in X.*

Proof. Suppose μ is a fuzzy first category set in X. Then $1-\mu$ is a fuzzy residual set in X. Since X is a fuzzy globally disconnected and fuzzy ∂^*, by Proposition 3.19, $cl(1-\mu)$ is a fuzzy F_σ-set in X. Thus $1 - int(\mu)$ is a fuzzy F_σ-set in X. So $int(\mu)$ is a fuzzy G_δ-set in X. □

Remark 3.22. From Corollary 3.21, one will have the following result:
If μ is a fuzzy first category set in a fuzzy globally disconnected and fuzzy ∂^* space (X, T), then there exists a fuzzy G_δ-set δ in X such that $\delta \leq \mu$.

Proposition 3.23. *If (X, T) is a fuzzy globally disconnected and fuzzy ∂^* space, then X is not a fuzzy Baire space.*

Proof. Suppose X is a fuzzy globally disconnected and fuzzy ∂^* space and let λ be a fuzzy residual set in X. Then by Proposition 3.19, $cl(\lambda)$ is a fuzzy F_σ-set in X. Thus $cl(\lambda) \neq 1$. So by Theorem 2.14, X is not a fuzzy Baire space. □

The following proposition gives a condition for a fuzzy globally disconnected and fuzzy ∂^* space to become a fuzzy Baire space.

Proposition 3.24. *If each fuzzy F_σ-set is a fuzzy dense set in a fuzzy globally disconnected and fuzzy ∂^* space (X, T), then X is a fuzzy Baire space.*

Proof. Let λ be a fuzzy residual set in X. Since X is a fuzzy globally disconnected and fuzzy ∂^* space, by Proposition 3.19, $cl(\lambda)$ is a fuzzy F_σ-set in X. Then by hypothesis, $cl[cl(\lambda)] = 1$. Thus $cl(\lambda) = 1$. So by Theorem 2.14, X is a fuzzy Baire space. □

4. Fuzzy $S^\star N$-spaces

Definition 4.1. A fuzzy topological space (X, T) is called a *fuzzy $S^\star N$-space*, if for each pair of fuzzy closed sets μ_1 and μ_2 in X with $\mu_1 \leq 1 - \mu_2$ there exist fuzzy simply* open sets $\lambda_1$and λ_2 in X such that $\mu_1 \leq \lambda_1$, $\mu_2 \leq \lambda_2$ and $\lambda_1 \leq 1 - \lambda_2$

Example 4.2. Let $X = \{a, b, c\}$. Consider the fuzzy sets α and β in X defined as follows:

$$\alpha(a) = 0.4,\ \alpha(b) = 0.5,\ \alpha(c) = 0.6,\ \beta(a) = 0.7,\ \beta(b) = 0.4,\ \beta(c) = 0.5.$$

Then $T = 0, \alpha, \beta, \alpha \vee \beta, \alpha \wedge \beta, 1$ is a fuzzy topology on X. By computation, one can see that $int(1-\alpha) = 0,\ int(1-\beta) = 0,\ int(1-[\alpha\vee\beta]) = 0,\ int(1-[\alpha\wedge\beta]) = \alpha\wedge\beta$ and $cl(\alpha) = 1,\ cl(\beta) = 1,\ cl(\alpha\vee\beta) = 1,\ cl(\alpha\wedge\beta) = 1-(\alpha\wedge\beta)$. By computation, one can see that fuzzy simply* open sets in (X, T) are $\alpha, \beta, (\alpha\vee\beta), (\alpha\wedge\beta), \{\alpha\vee[1-\alpha]\}, \{\beta\vee$

$[1-\alpha]\}, \{(\alpha\wedge\beta)\vee[1-\alpha]\}, \{\alpha\vee[1-\beta]\}, \{\beta\vee[1-\beta]\}, \{(\alpha\wedge\beta)\vee[1-\beta]\}, \{(\alpha\wedge\beta)\vee[1-\beta]\}$ and $\{(\alpha \wedge \beta) \vee [1 - [\alpha \vee \beta]]\}$.

Now $\{(1 - \alpha), (1 - [\alpha \vee \beta])\}$ is the only pair of fuzzy closed sets in (X, T) with $(1 - \alpha) \leq 1 - 1 - [\alpha \vee \beta]$. For the fuzzy closed sets $(1 - \alpha)$ and $(1 - [\alpha \vee \beta])$, there exist fuzzy simply* open sets $(\alpha \wedge \beta) \vee [1 - \alpha]$ and $\{(\alpha\wedge\beta)\vee[1-[\alpha\vee\beta]]\}$ in X such that $(1 - \alpha) \leq \{(\alpha \wedge \beta) \vee [1 - \alpha]\}$ and $(1 - [\alpha \vee \beta]) \leq \{(\alpha \wedge \beta) \vee [1 - [\alpha \vee \beta]\}$. Also, $\{(\alpha\wedge\beta)\vee[1-\alpha]\} \leq 1-\{(\alpha\wedge\beta)\vee[1-[\alpha\vee\beta]]\}$. Thus (X, T) is a fuzzy $S^\star N$- space.

It should be noted that for the fuzzy closed sets $(1 - \alpha)$ and $(1 - [\alpha \vee \beta])$, there exist fuzzy open sets $\alpha\vee\beta$ and α in X such that $(1-\alpha) \leq \alpha\vee\beta$ and $(1-[\alpha\vee\beta]) \leq \alpha$. But $(\alpha\vee\beta) > 1-\alpha$. That is, $(\alpha\vee\beta) \nleq 1-\alpha$. So (X, T) is not a fuzzy normal space.

Example 4.3. Let $X = \{a, b, c\}$. Consider the fuzzy sets λ, μ and γ in X defined as follows:

$$\begin{aligned} &\lambda(a) = 0.7, \quad \lambda(b) = 0.5, \quad \lambda(c) = 0.6,\\ &\mu(a) = 0.4, \quad \mu(b) = 0.7, \quad \mu(c) = 0.5,\\ &\gamma(a) = 0.6, \quad \gamma(b) = 0.4, \quad \gamma(c) = 0.8. \end{aligned}$$

Then $T = \{0, \lambda, \mu, \gamma, \lambda \vee \mu, \lambda \vee \gamma, \lambda \wedge \mu, \lambda \wedge \gamma, \mu \wedge \gamma, \mu \vee [\lambda \wedge \gamma],\ \gamma \vee [\lambda \wedge \mu], \lambda \wedge [\mu \vee \gamma], \lambda \vee \mu, 1\}$ is a fuzzy topology on X. By computation, one can find that $\lambda, \mu, \gamma, \lambda \vee \mu, \lambda \vee \gamma, \lambda \wedge \gamma, \gamma \vee [\lambda \wedge \mu], \lambda \wedge [\mu \vee \gamma]$ are fuzzy dense sets in (X, T) and $cl(\lambda \wedge \mu) = 1 - (\lambda \wedge \mu)$, $cl(\lambda \wedge \gamma) = 1 - (\lambda \wedge \gamma)$. Also, $\text{int}(1 - (\lambda \wedge \mu)) = \lambda\wedge\mu$, $\text{int}(1-(\mu\wedge\gamma)) = \lambda\wedge\mu$. The fuzzy nowhere dense sets in (X, T) are $1-\lambda, 1-\mu, 1-\gamma, 1-(\lambda\vee\mu), 1-(\lambda\vee\gamma), 1-(\mu\vee\gamma), 1-(\lambda\wedge\gamma), 1-(\mu\wedge\gamma), 1-(\lambda\wedge\mu), 1-(\mu\vee[\lambda\wedge\gamma]), \gamma\vee[\lambda\wedge\mu], \lambda\wedge[\mu\vee\gamma], \lambda\wedge[\mu\vee\gamma], \lambda\wedge[\mu\vee\gamma]$. By computation, one can find that $\lambda, \mu, \gamma, \lambda \vee \mu, \lambda \vee \gamma, \mu \vee \gamma, \lambda \wedge \gamma, \mu \wedge \lambda, \mu \vee [\lambda \wedge \gamma], \gamma \vee [\lambda \wedge \mu], \lambda \wedge [\mu \vee \gamma]$ are fuzzy simply* open sets in (X, T).

Now, for the fuzzy closed sets $(1 - [\lambda \wedge \mu])$ and $(1 - [\mu \wedge \gamma])$, there exist fuzzy simply* open sets $\{\mu\wedge\gamma\}$ and $\{(\lambda\wedge\gamma)\vee(1-\gamma)\}$ in (X, T) such that $(1-[\lambda\wedge\mu]) \leq (\mu\wedge\gamma)$ and $(1-[\mu\wedge\gamma]) \leq ((\lambda\wedge\gamma)\vee(1-\gamma))$. But, $\{\mu\wedge\gamma\} \nleq 1-\{(\lambda\wedge\gamma)\vee(1-\gamma)\}$. Thus (X, T) is not a fuzzy $S^* N$-space.

Proposition 4.4. *If μ_1 and μ_2 are any two fuzzy closed sets in a fuzzy $S^\star N$-space (X, T) with $\mu_1 \leq 1 - \mu_2$, then there exist fuzzy simply* open sets λ_1 and λ_2 in X such that $\mu_1 \leq \lambda_1 \leq 1 - \lambda_2 \leq 1 - \mu_2$ and $cl(\lambda_1) \neq 1$ and $cl(\lambda_2) \neq 1$.*

Proof. Suppose μ_1 and μ_2 are any two fuzzy closed sets in a fuzzy $S^\star N$-space X with $\mu_1 \leq 1 - \mu_2$. Then there exist fuzzy simply* open sets λ_1 and λ_2 in X such that $\mu_1 \leq \lambda_1$, $\mu_2 \leq \lambda_2$ and $\lambda_{1\leq 1-\lambda_2}$. Now $\mu_2 \leq \lambda 2$ implies that $1 - \lambda_2 \leq 1 - \mu_2$ and thus $\mu_1 \leq \lambda_1 \leq 1 - \lambda_2 \leq 1 - \mu_2$. So for the fuzzy simply* open sets λ_1 and λ_2 with $\lambda_1 \leq 1 - \lambda_2$, by Proposition 3.15, $cl(\lambda_1) \neq 1$ and $cl(\lambda_2) \neq 1$, in (X, T), □

Corollary 4.5. *If μ_1 and μ_2 are any two fuzzy closed sets in a fuzzy $S^* N$-space (X, T) with $\mu_1 \leq 1 - \mu_2$, then there exist a fuzzy simply* open set δ and a fuzzy simply* closed set γ in X such that $\mu_1 \leq \delta \leq \gamma \leq 1 - \mu_2$.*

Proof. Suppose μ_1 and μ_2 are any two fuzzy closed sets in a fuzzy $S^* N$-space X with $\mu_1 \leq 1 - \mu_2$. Then by Proposition 4.4, there exist fuzzy simply* open sets λ_1 and λ_2 in (X, T) such that $\mu_1 \leq \lambda_1 \leq 1 - \lambda_2 \leq 1 - \mu_2$. Let $\delta = \lambda_1$ and $\gamma = 1 - \lambda_2$. Then for the fuzzy closed sets μ_1 and μ_2, there exist a fuzzy simply* open set δ and a fuzzy simply* closed set γ in X such that $\mu_1 \leq \delta \leq \gamma \leq 1 - \mu_2$. □

Proposition 4.6. *If $\mu \leq \lambda$, where μ is a fuzzy closed set and λ is a fuzzy open set in a fuzzy S^*N-space (X,T), then there exist a fuzzy simply* open set δ and a fuzzy simply* closed set γ in X such that $\mu \leq \delta \leq \gamma \leq \lambda$.*

Proof. Suppose $\mu \leq \lambda$, where μ is a fuzzy closed set and λ is a fuzzy open set in X. Then $\mu \leq 1-(1-\lambda)$ and $\mu, 1-\lambda$ are fuzzy closed sets in X. Since X is a fuzzy S^*N-space, by Corollary 4.5, there exist a fuzzy simply* open set δ and a fuzzy simply* closed set γ in X such that $\mu \leq \delta \leq \gamma \leq 1-(1-\lambda)$. Thus $\mu \leq \delta \leq \gamma \leq \lambda$. □

Corollary 4.7. *If $\mu \leq \lambda$, where μ is a fuzzy closed set and λ is a fuzzy open set in a fuzzy S^*N-space (X,T) with $\mu_1 \leq 1-\mu_2$, then there exist a fuzzy simply* open set δ in X such that $\mu \leq \delta \leq \lambda$.*

Proposition 4.8. *If μ_1 and μ_2 are any two fuzzy closed sets in a fuzzy S^*N-space (X,T) with $\mu_1 \leq 1-\mu_2$, then there exist a fuzzy G_δ-set θ in X such that $int(\mu_1) \leq \theta \leq 1-int(\mu_2)$.*

Proof. Suppose μ_1 and μ_2 are any two fuzzy closed sets in a fuzzy S^*N-space X with $\mu_1 \leq 1-\mu_2$. Then by Corollary 4.5, there exist a fuzzy simply* open set δ and a fuzzy simply* closed set γ in X such that $\mu_1 \leq \delta \leq \gamma \leq 1-\mu_2$. Now $\mu_1 \leq \gamma \leq 1-\mu_2$ implies that $int(\mu_1) \leq int(\gamma) \leq int(1-\mu_2)$ and thus $int(\mu_1) \leq int(\gamma) \leq 1-cl(\mu_2) = 1-\mu_2 \leq 1-int(\mu_2)$. By Proposition 3.6, for the fuzzy simply* closed set γ, $int(\gamma)$ is a fuzzy G_δ-set in X. Let $\theta = int(\gamma)$. Then it follows that $int(\mu_1) \leq \theta \leq 1-int(\mu_2)$. □

Proposition 4.9. *If μ_1 and μ_2 are any two fuzzy closed sets in a fuzzy S^*N-space (X,T) with $\mu_1 \leq 1-\mu_2$, then there exists a fuzzy F_σ-set ρ in X such that $\mu_1 \leq \rho \leq 1-int(\mu_2)$.*

Proof. Suppose μ_1 and μ_2 are any two fuzzy closed setsin a fuzzy S^*N-space X with $\mu_1 \leq 1-\mu_2$. Then by Corollary 4.5, there exist a fuzzy simply* open set δ and a fuzzy simply* closed set γ in X such that $\mu_1 \leq \delta \leq \gamma \leq 1-\mu_2$. Now $\mu_1 \leq \delta \leq 1-\mu_2$ implies that $cl(\mu_1) \leq cl(\delta) \leq cl(1-\mu_2)$ and thus $\mu_1 \leq cl(\mu_1) \leq cl(\delta) \leq 1-int(\mu_2)$. By Proposition 3.4, for the fuzzy simply* open set δ, $cl(\delta)$ is a fuzzy F_σ-set in X. Let $\rho = cl(\delta)$. Then it follows that $\mu_1 \leq \rho \leq 1-int(\mu_2)$. □

Proposition 4.10. *If μ_1 and μ_2 are any two fuzzy closed sets in a fuzzy S^*N-space (X,T) with $\mu_1 \leq 1-\mu_2$, then there exist fuzzy somewhere dense sets λ_1 and λ_2 in X such that $\mu_1 \leq \lambda_1$, $\mu_2 \leq \lambda_2$ with $clint(\lambda_1) \neq 1$ and $clint(\lambda_2) \neq 1$.*

Proof. Suppose μ_1 and μ_2 are any two fuzzy closed sets in a fuzzy S^*N-space X with $\mu_1 \leq 1-\mu_2$. Since X is a fuzzy S^*N-space, there exist fuzzy simply* open sets λ_1 and λ_2 in X such that $\mu_1 \leq \lambda_1$, $\mu_2 \leq \lambda_2$ and $\lambda_1 \leq 1-\lambda_2$. By Theorem 2.11, for the fuzzy simply* open sets λ_1 and λ_2, $int(\lambda_1) \neq 0$ and $int(\lambda_2) \neq 0$. Now $int(\lambda_1) \leq intcl(\lambda_1)$ and $int(\lambda_2) \leq intcl(\lambda_2)$, implies that $intcl(\lambda_1) \neq 0$ and $intcl(\lambda_2) \neq 0$. Thus the fuzzy simply* open sets λ_1 and λ_2 are fuzzy somewhere dense sets in X. So for the fuzzy closed sets μ_1 and μ_2, there exist fuzzy somewhere dense sets λ_1 and λ_2 in X such that $\mu_1 \leq \lambda_1$, $\mu_2 \leq \lambda_2$ and $\lambda_1 \leq 1-\lambda_2$. Now $intcl(\lambda_1) \leq intcl(1-\lambda_2)$ implies that $intcl(1-\lambda_2) \neq 0$. By Lemma 2.3, $1-clint(\lambda_2) \neq 0$ and $clint(\lambda_2) \neq 1$. Also, $\lambda_1 \leq 1-\lambda_2$ implies that $\lambda_2 \leq 1-\lambda_1$. Hence $clint(\lambda_1) \neq 1$. □

Proposition 4.11. *If δ_1 and δ_2 are any two fuzzy open sets in a fuzzy S^*N-space (X,T) such that $1-\delta_1 \leq \delta_2$, then there exist fuzzy simply*-closed sets γ_1 and γ_2 in X such that $\gamma_1 \leq \delta_1$, $\gamma_2 \leq \delta_2$ and $1-\gamma_1 \leq \gamma_2$.*

Proof. Suppose δ_1 and δ_2 are any two fuzzy open sets in a fuzzy S^*N-space X such that $1-\delta_1 \leq \delta_2$. Then $1-\delta_1$ and $1-\delta_2$ are fuzzy closed sets in X such that $1-\delta_1 \leq 1-(1-\delta_2)$. Thus there exist fuzzy simply* open sets λ_1 and λ_2 in X such that $1-\delta_1 \leq \lambda_1$, $1-\delta_2 \leq \lambda_2$ and $\lambda_1 \leq 1-\lambda_2$. This implies that $1-\lambda_1 \leq \delta_1$ and $1-\lambda_2 \leq \delta_2$. Let $\gamma_1 = 1-\lambda_1$ and $\gamma_2 = 1-\lambda_2$. Then γ_1 and γ_2 are fuzzy simply*-closed sets in X such that $\gamma_1 \leq \delta_1$ and $\gamma_2 \leq \delta_2$. Now $\lambda_1 \leq 1-\lambda_2$ implies that $1-\gamma_1 \leq 1-(1-\gamma_2)$ and thus $1-\gamma_1 \leq \gamma_2$. □

Proposition 4.12. *If δ_1 and δ_2 are fuzzy open sets in a fuzzy S^*N-space (X,T) such that $\delta_1 \vee \delta_2 = 1$, then there exist fuzzy simply* open sets λ_1 and λ_2 in X such that $1-\delta_1 \leq \lambda_1 \leq 1-\lambda_2 \leq \delta_2$.*

Proof. Suppose δ_1 and δ_2 are two fuzzy open sets in a fuzzy S^*N-space X such that $\delta_1 \vee \delta_2 = 1$. Then $1-(\delta_1 \vee \delta_2) = 0$, i.e., $(1-\delta_1) \wedge (1-\delta_2) = 0$. This implies that $1-\delta_1 \leq 1-(1-\delta_2)$. Thus $1-\delta_1$ and $1-\delta_2$ are fuzzy closed sets in the fuzzy S^*N-space X such that $1-\delta_1 \leq 1-(1-\delta_2)$. So there exist fuzzy simply*-open sets λ_1 and λ_2 in X such that $1-\delta_1 \leq \lambda_1$, $1-\delta_2 \leq \lambda_2$ and $\lambda_1 \leq 1-\lambda_2$. This implies that $1-\delta_1 \leq \lambda_1 \leq 1-\lambda_2 \leq \delta_2$. □

Corollary 4.13. *If δ_1 and δ_2 are fuzzy open sets in a fuzzy S^*N-space (X,T) such that $\delta_1 \vee \delta_2 = 1$, then there exist a fuzzy simply*-open set λ and a fuzzy simply*-closed set μ in X such that $1-\delta_1 \leq \lambda \leq \mu \leq \delta_2$.*

Proposition 4.14. *If μ_1 and μ_2 are any two fuzzy closed sets in a fuzzy S^*N-space (X,T) with $\mu_1 \leq 1-\mu_2$, then there exist fuzzy regular open sets α and β in X such that $int(\mu_1) \leq \alpha$ and $int(\mu_2) \leq \beta$.*

Proof. Suppose μ_1 and μ_2 are any two fuzzy closed sets in a fuzzy S^*N-space X with $\mu_1 \leq 1-\mu_2$. Then there exist fuzzy simply* open sets λ_1 and λ_2 in X such that $\mu_1 \leq \lambda_1$, $\mu_2 \leq \lambda_2$ and $\lambda_1 \leq 1-\lambda_2$. By Proposition 3.3, for the fuzzy simply* open sets λ_1 and λ_2, there exist fuzzy regular closed sets θ_1 and θ_2 in (X,T) such that $\theta_1 \leq clint(\lambda_1)$ and $\theta_2 \leq clint(\lambda_2)$. Now $\lambda_1 \leq 1-\lambda_2$ implies that $clint(\lambda_1) \leq clint(1-\lambda_2)$ and thus $\theta_1 \leq clint(1-\lambda_2) = 1-intcl(\lambda_2)$. So $intcl(\lambda_2) \leq 1-\theta_1$. Let $\beta = 1-\theta_1$. Then β is a fuzzy regular open set in (X,T). Now $\mu_2 \leq \lambda_2$ implies that $intcl(\mu_2) \leq intcl(\lambda_2) \leq \beta$ and thus $int(\mu_2) \leq \beta$, in (X,T). Also, $\lambda_1 \leq 1-\lambda_2$ implies that $\lambda_2 \leq 1-\lambda_1$. So $clint(\lambda_2) \leq clint(1-\lambda_1)$ and $\theta_2 \leq clint(\lambda_2) \leq 1-intcl(\lambda_1)$. It follows that $intcl(\lambda_1) \leq 1-\theta_2$. Let $\alpha = 1-\theta_2$. Then α is a fuzzy regular open set in X. Since $\mu_1 \leq \lambda_1$, $intcl(\mu_1) \leq intcl(\lambda_1) \leq \alpha$. Thus $int(\mu_1) \leq \alpha$. □

5. Fuzzy S^*N-spaces and other fuzzy topological spaces

Proposition 5.1. *If a fuzzy topological space (X,T) is a fuzzy normal space, then X is a fuzzy S^*N-space.*

Proof. Suppose X is a fuzzy normal space and let μ_1 and μ_2 be any two fuzzy closed sets with $\mu_1 \leq 1-\mu_2$. Then by the hypothesis, there exist fuzzy open sets λ_1 and λ_2

in X such that $\mu_1 \leq \lambda_1$, $\mu_2 \leq \lambda_2$ and $\lambda_1 \leq 1-\lambda_2$. Since each fuzzy open set is a fuzzy simply*-open set in a fuzzy topological space (Remarks 3.1(1) in [7]), λ_1 and λ_2 are fuzzy simply*-open sets in X. Thus it follows that X is a fuzzy S^*N-space. □

It is to be noted that fuzzy S^*N-spaces need not be fuzzy normal spaces. For, in example, 4.2, (X,T) is a fuzzy S^*N-space, but not a fuzzy normal space. It is very natural to ask under which conditions does a fuzzy S^*N-space become a fuzzy normal space? The following proposition provides an answer to this question.

Proposition 5.2. *If a fuzzy topological space (X,T) is a fuzzy perfectly disconnected and fuzzy S^*N-space, then X is a fuzzy normal space.*

Proof. Let μ_1 and μ_2 be any two fuzzy closed sets with $\mu_1 \leq 1-\mu_2$. Since X is a fuzzy S^*N-space, there exist fuzzy simply* open sets λ_1 and λ_2 in X such that $\mu_1 \leq \lambda_1$, $\mu_2 \leq \lambda_2$ and $\lambda_1 \leq 1-\lambda_2$. Since X is a fuzzy perfectly disconnected space, by Proposition 3.10, the fuzzy simply* open sets λ_1 and λ_2 are fuzzy open sets in X. Then it follows that (X,T) is a fuzzy normal space. □

It is to be noted that fuzzy semi normal spaces need not be fuzzy S^*N-spaces. Consider the following example.

Example 5.3. Let $X = \{a,b,c\}$. Consider the fuzzy sets α,β,γ and δ in X defined as follows:

$$\alpha(a) = 0.4,\ \alpha(b) = 0.6,\ \alpha(c) = 0.4,\ \beta(a) = 0.6,\ \beta(b) = 0.5,\ \beta(c) = 0.6,$$

$$\gamma(a) = 0.6,\ \gamma(b) = 0.5,\ \gamma(c) = 0.7,\ \delta(a) = 0.5,\ \delta(b) = 0.6,\ \delta(c) = 0.5.$$

Then $T = \{0,\alpha,\beta,\gamma,\alpha\vee\beta,\alpha\vee\gamma,\alpha\vee\beta,1\}$ is a fuzzy topology on X. By computation, one can see that int $(1-\alpha) = 0$; $int(1-\beta) = \alpha\wedge\beta$; $int(1-\gamma) = 0$, $int(1-[\alpha\vee\beta]) = 0$, $int(1-[\alpha\vee\gamma]) = 0$, $int(1-[\alpha\wedge\beta]) = \beta$ and $int(1-\delta) = 0$, $cl(\beta) = 1-(\alpha\wedge\beta)$, $cl(\alpha\wedge\beta) = 1-\beta$, $cl(\delta) = 1$, $cl(1-\delta) = 1-\alpha$. The fuzzy dense sets in (X,T) are α, γ, $\alpha\vee\beta$ and $\alpha\vee\gamma$ and the fuzzy nowhere dense sets in (X,T) are $1-\alpha$, $1-\gamma$, $1-[\alpha\vee\beta]$, $1-[\alpha\vee\gamma]$ and $1-\delta$.

Now $intcl(\beta) = int(1-[\alpha\wedge\beta]) = \beta$ and $intcl(\alpha\wedge\beta) = int(1-\beta) = \alpha\wedge\beta$ and thus β and $\alpha\wedge\beta$ are the fuzzy regular open sets in (X,T). For each fuzzy closed set $\lambda(= 1-\alpha, 1-\beta, 1-\gamma, 1-[\alpha\vee\beta], 1-[\alpha\vee\gamma], 1-[\alpha\wedge\beta])$ and each fuzzy open set $\mu(=\alpha,\beta,\gamma,\alpha\vee\beta,\alpha\vee\gamma,\alpha\wedge\beta)$ such that $\lambda\leq\mu$, there exists a fuzzy regular open set $\sigma(=\beta$ or $\alpha\wedge\beta)$ such that $\lambda\leq\sigma\ \leq\mu$ implies that the fuzzy topological space (X,T) is a fuzzy semi normal space.

The fuzzy *simply**open sets in (X,T) are $\alpha,\beta,\gamma,\alpha\vee\beta,\alpha\vee\gamma,\alpha\wedge\beta,\{\alpha\vee(1-\delta)\},\{(\alpha\wedge\beta)\vee(1-\delta))\}$. For the fuzzy closed sets $\{1-[\alpha\wedge\beta]\}$ and $(1-\gamma)$ in (X,T), there exist fuzzy *simply**open sets $\lambda_1(=\gamma,\alpha\vee\beta,\alpha\vee\gamma)$ and $\lambda_2(=\alpha,\beta,\gamma,\alpha\vee\beta,\alpha\vee\gamma,\alpha\wedge\beta\{\alpha\vee(1-\delta)\},\{(\alpha\wedge\beta)\vee(1-\delta)\})$ in (X,T) such that $\{1-[\alpha\wedge\beta]\}\leq\lambda_1,(1-\gamma)\leq\lambda_2$. But $\lambda_1\nleq 1-\lambda_2$ implies that (X,T) is not a fuzzy S^*N-space.

The following proposition gives a condition for fuzzy semi normal spaces to become fuzzyS^*N-spaces.

Proposition 5.4. *If a fuzzy topological space (X,T) is a fuzzy semi normal and fuzzy extremally disconnected space, then (X,T) is a fuzzy S^*N-space.*

Proof. Let μ_1 and μ_2 be any two fuzzy closed sets in X with $\mu_1 \leq 1-\mu_2$. Since X is a fuzzy semi normal space, for the fuzzy closed set μ_1 and the fuzzy open set $1-\mu_2$ such that $\mu_1 \leq 1-\mu_2$, there exists a fuzzy regular open set σ in X such that $\mu_1 \leq \sigma \leq 1-\mu_2$. Now, $\mu_1 \leq \sigma$ implies that $1-\mu_1 \geq 1-\sigma$, where $1-\mu_1$ is a fuzzy open set and $1-\sigma$ is a fuzzy regular closed set in (X,T). Since a fuzzy regular closed set is a fuzzy closed set in a fuzzy topological space, $1-\sigma$ is a fuzzy closed set in X. Again, X being a fuzzy semi normal space, for the fuzzy closed set $1-\sigma$ and the fuzzy open set $1-\mu_1$ such that $1-\sigma \leq 1-\mu_1$, there exists a fuzzy regular open set γ in X such that $1-\sigma \leq \gamma \leq 1-\mu_1$. This implies that $\mu_1 \leq 1-\gamma \leq \sigma$. Also, $\sigma \leq 1-\mu_2$ implies that $\mu_2 \leq 1-\sigma$. Let $\lambda_1 = 1-\gamma$ and $\lambda_2 = 1-\sigma$. Then λ_1 and λ_2 are fuzzy regular closed sets in X such that $\mu_1 \leq \lambda_1, \mu_2 \leq \lambda_2$ and $\lambda_1 \leq 1-\lambda_2$.

Now, λ_1 and λ_2 are fuzzy regular closed sets in X imply that $clint(\lambda_1) = \lambda_1$ and $clint(\lambda_2) = \lambda_2$. Since X is a fuzzy extremally disconnected space, for the fuzzy open sets $int(\lambda_1)$, $int(\lambda_2)$ in X, $clint(\lambda_1)$ and $clint(\lambda_2)$ are fuzzy open sets in X and thus λ_1 and λ_2 are fuzzy open sets in X. Since each fuzzy open set is a fuzzy simply* open set in a fuzzy topological space, λ_1 and λ_2 are fuzzy simply* open sets in X. So for the fuzzy closed sets μ_1 and μ_2, there exist fuzzy simply* open sets λ_1 and λ_2 such that $\mu_1 \leq \lambda_1$, $\mu_2 \leq \lambda_2$ and $\lambda_1 \leq 1-\lambda_2$. Hence X is a fuzzy S^*N-space. □

The following proposition gives a condition for fuzzy D-Baire spaces to become fuzzy S^*N-spaces.

Proposition 5.5. *If there exist fuzzy pseudo-open sets λ_1 and λ_2 such that $\mu_1 \leq \lambda_1$, $\mu_2 \leq \lambda_2$ and $\lambda_1 \leq 1-\lambda_2$, for any two fuzzy closed sets μ_1 and μ_2 with $\mu_1 \leq 1-\mu_2$, in a fuzzy D-Baire space (X,T), then (X,T) is a fuzzy S^*N-space.*

Proof. Let μ_1 and μ_2 be any two fuzzy closed sets in X with $\mu_1 \leq 1-\mu_2$. By the hypothesis, there exist fuzzy pseudo-open sets λ_1 and λ_2 in X such that $\mu_1 \leq \lambda_1$, $\mu_2 \leq \lambda_2$ and $\lambda_1 \leq 1-\lambda_2$. Since X is a fuzzy D-Baire space, by Theorem 2.8, the fuzzy pseudo-open sets λ_1 and λ_2 are fuzzy simply*-open sets in X and it follows that X is a fuzzy S^*N-space. □

Proposition 5.6. *If $\mu \leq \lambda$, where μ is a fuzzy closed set and λ is a fuzzy open set in a fuzzy nodef and fuzzy S^*N-space (X,T), then there exist a fuzzy pseudo-open set δ in X such that $\mu \leq \delta \leq \lambda$.*

Proof. Suppose $\mu \leq \lambda$, where μ is a fuzzy closed set and λ is a fuzzy open set in X. Since X is a fuzzy S^*N-space, by Corollary 4.7, there exists a fuzzy simply* open set δ in X such that $\mu \leq \delta \leq \lambda$. Since X is a fuzzy nodef space, fuzzy nowhere dense sets are F_σ-sets in X. Then δ is a fuzzy simply* open set in X in which fuzzy nowhere dense sets are F_σ-sets. Thus by Theorem 2.7, the fuzzy simply* open set δ is a fuzzy pseudo-open set in X. □

Proposition 5.7. *If μ_1 and μ_2 are any two fuzzy closed sets with $\mu_1 \leq 1-\mu_2$, in a fuzzy nodef and fuzzy S^*N-space (X,T), then there exist fuzzy pseudo-open sets λ_1 and λ_2 in X such that $\mu_1 \leq \lambda_1$, $\mu_2 \leq \lambda_2$ and $\lambda_1 \leq 1-\lambda_2$.*

Proof. Suppose μ_1 and μ_2 are any two fuzzy closed setsin X with $\mu_1 \leq 1-\mu_2$. Since X is a fuzzy S^*N-space, there exist fuzzy simply* open sets λ_1 and λ_2 in X such that $\mu_1 \leq \lambda_1$, $\mu_2 \leq \lambda_2$ and $\lambda_1 \leq 1 - \lambda_2$. Since X is a fuzzy nodef space, fuzzy nowhere dense sets are F_σ-sets in X. Then λ_1 and λ_2 are fuzzy simply* open sets in X in which fuzzy nowhere dense sets are F_σ-sets. Thus by Theorem 2.7, the fuzzy simply* open sets λ_1 and λ_2 are fuzzy pseudo-open sets in X. □

Proposition 5.8. *If μ_1 and μ_2 are any two fuzzy closed sets with $\mu_1 \leq 1 - \mu_2$, in a fuzzy perfectly disconnected and fuzzy S^*N-space (X, T), then there exist fuzzy F_σ-sets δ_1 and δ_2 in X such that $\mu_1 \leq \delta_1$, $\mu_2 \leq \delta_2$ and $\delta_1 \leq 1 - \delta_2$.*

Proof. Suppose μ_1 and μ_2 are any two fuzzy closed sets in X with $\mu_1 \leq 1-\mu_2$. Since X is a fuzzy S^*N-space, there exist fuzzy simply* open sets λ_1 and λ_2 in X such that $\mu_1 \leq \lambda_1$, $\mu_2 \leq \lambda_2$ and $\lambda_1 \leq 1 - \lambda_2$. Since X is a fuzzy perfectly disconnected space, for the fuzzy sets λ_1 and λ_2 with $\lambda_1 \leq 1 - \lambda_2$, $\mathrm{cl}(\lambda_1) \leq 1 - \mathrm{cl}(\lambda_2)$, in (X, T). By Proposition 3.4, for the fuzzy simply* open sets λ_1 and λ_2, $\mathrm{cl}(\lambda_1)$ and $\mathrm{cl}(\lambda_2)$ are fuzzy F_σ-sets in X. Now $\mu_1 \leq \lambda_1$ implies that $\mathrm{cl}(\mu_1) \leq \mathrm{cl}(\lambda_1)$ and then $\mu_1 \leq \mathrm{cl}(\lambda_1)$, and $\mu_2 \leq \lambda_2$ implies that $\mathrm{cl}(\mu_2) \leq \mathrm{cl}(\lambda_2)$ and then $\mu_2 \leq \mathrm{cl}(\lambda_2)$. Let $\delta_1 = \mathrm{cl}(\lambda_1)$ and $\delta_2 = \mathrm{cl}(\lambda_2)$. Thus for the fuzzy closed sets μ_1 and μ_2, there exist fuzzy F_σ-sets δ_1 and δ_2 in X such that $\mu_1 \leq \delta_1$, $\mu_2 \leq \delta_2$ and $\delta_1 \leq 1 - \delta_2$. □

The following two propositions give conditions for fuzzy ∂^* spaces to become fuzzy S^*N-spaces.

Proposition 5.9. *If there exist fuzzy co-σ-boundary sets λ_1 and λ_2 such that $\mu_1 \leq \lambda_1$, $\mu_2 \leq \lambda_2$ and $\lambda_1 \leq 1-\lambda_2$, for any two fuzzy closed sets μ_1 and μ_2 with $\mu_1 \leq 1-\mu_2$, in a fuzzy ∂^* space (X, T), then X is a fuzzy S^*N-space.*

Proof. Let μ_1 and μ_2 be any two fuzzy closed sets in X with $\mu_1 \leq 1 - \mu_2$. By the hypothesis, there exist fuzzy co-σ-boundary sets λ_1 and λ_2 in X such that $\mu_1 \leq \lambda_1$, $\mu_2 \leq \lambda_2$ and $\lambda_1 \leq 1 - \lambda_2$. Since X is a fuzzy ∂^* space, by Theorem 2.12, the fuzzy co-σ-boundary sets λ_1 and λ_2 are fuzzy simply*-open sets in X and it follows that (X, T) is a fuzzy S^*N-space. □

Proposition 5.10. *If there exist fuzzy residual sets λ_1 and λ_2 such that $\mu_1 \leq \lambda_1$, $\mu_2 \leq \lambda_2$ and $\lambda_1 \leq 1-\lambda_2$, for any two fuzzy closed sets μ_1 and μ_2 with $\mu_1 \leq 1-\mu_2$, in a fuzzy globally disconnected and fuzzy ∂^* space (X, T), then X is a fuzzy S^*N-space.*

Proof. Let μ_1 and μ_2 be any two fuzzy closed sets in X with $\mu_1 \leq 1 - \mu_2$. By hypothesis, there exist fuzzy residual sets λ_1 and λ_2 in X such that $\mu_1 \leq \lambda_1$, $\mu_2 \leq \lambda_2$ and $\lambda_1 \leq 1 - \lambda_2$. Since X is a fuzzy globally disconnected and fuzzy ∂^* space, by Theorem 2.13, the fuzzy residual sets λ_1 and λ_2 are fuzzy simply*-open sets in X and it follows that (X, T) is a fuzzy S^*N-space. □

6. Conclusion

In this paper, it is established that the fuzzy closure of a fuzzy simply* open set is a fuzzy F_σ-set and the fuzzy interior of a fuzzy simply* closed set is a fuzzy G_δ-set in fuzzy topological spaces. It is obtained that fuzzy simply* open sets coincide with fuzzy open sets in fuzzy perfectly disconnected spaces. It is established that disjoint fuzzy simply* open sets are not fuzzy dense sets in fuzzy topological spaces. It is found that the fuzzy closure of a fuzzy G_δ-set is a fuzzy F_σ-set and the fuzzy interior of a fuzzy F_σ-set is a fuzzy G_δ-set in fuzzy ∂^* spaces. Also, it is proved that the fuzzy closure of a fuzzy residual set is a fuzzy F_σ-set and the fuzzy interior of a fuzzy first category set is a fuzzy G_δ-set in fuzzy globally disconnected and fuzzy ∂^* spaces. A condition for fuzzy globally disconnected and fuzzy ∂^* spaces to become fuzzy Baire spaces is also obtained.

Also, the notion of fuzzy S^*N-spaces is introduced by means of fuzzy simply* open sets and studied. Several characterizations of fuzzy S^*N-spaces are obtained. It is shown by examples that fuzzy S^*N-spaces need not be fuzzy normal spaces and fuzzy seminormal spaces need not be fuzzy S^*N-spaces. It is established that fuzzy perfectly disconnected and fuzzy S^*N-spaces are fuzzy normal spaces and fuzzy seminormal spaces with fuzzy extremally disconnectedness are fuzzy S^*N-spaces. The conditions, under which fuzzy ∂^* spaces, fuzzy D-Baire spaces become fuzzy S^*N-spaces, are also obtained in this paper.

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